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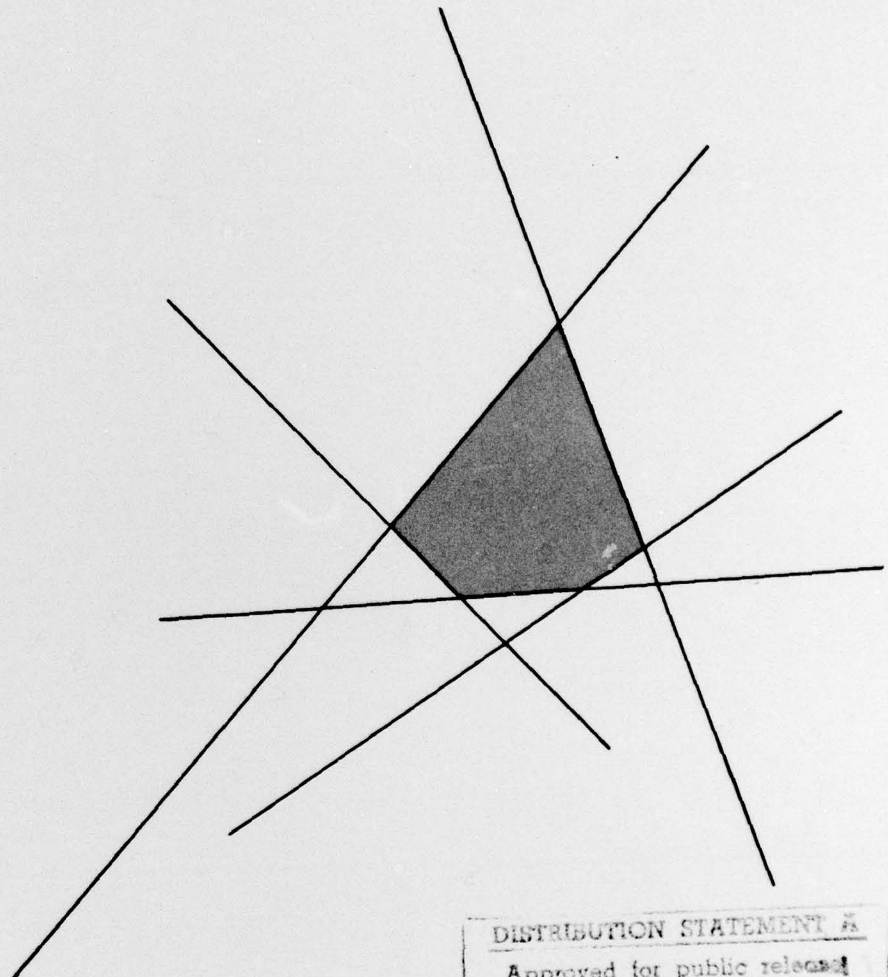
by
DAVID GALE



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by

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ABSTRACT

It is shown that the problem of finding a point in the intersection of a finite number of balls in n-space is equivalent to solving a very simple (one constraint) quadratic program.

SOLUTIONS OF SPHERICAL INEQUALITIES

by

David Gale

There are many examples of natural geometrical problems which can be solved by linear or quadratic programming. For example the problem of finding the largest ball contained in a given convex polytope (the *inscribed ball*) turns out to be a linear program, while the problem of finding the smallest ball containing a convex polytope (the *circumscribed ball*) is equivalent to a quadratic program [1]. In this note we extend the result of [1] by showing that the following problem can be solved by a simple quadratic program:

Problem:

Given a finite set of balls in R^n decide whether their intersection is nonempty and if so find a point in it.

To formulate the problem algebraically let c_i be the center, ρ_i the radius of the i^{th} ball, $i = 1, \ldots, n$.

Problem:

Solve the system of inequalities

(1) $$(x - c_i)^2 \leq \rho_i^2 {}^{\dagger}, \quad i = 1, \ldots, m ,$$

or show that no solution exists.

An obvious modification of this problem is to find a vector x and number z such that

---

† In this paper the scalar product of vectors x and y is written xy the scalar product of x with itself is x^2 .

(2) z is a maximum

subject to

(3) $$(x - c_i)^2 - \rho_i^2 + z \leqq 0 \ , \ i = 1, \ldots, m \ .$$

Letting $(\bar{x}, \bar{z})$ be the solution of (2), (3), it is clear that (1) has a solution if and only if $\bar{z} \geqq 0$. For the special case $\rho_i = 0$ for all i it is also clear that $\bar{z}$ is negative (unless all c_i are equal) in which case, by definition, $\bar{x}$ is the center and $-\bar{z}$ is the radius of the circumscribed ball.

The point of this note is to show that the problem (2), (3) has a dual which is a very simple quadratic program and hence can be solved by well known techniques. It then follows that if the c_i and ρ_i are rational then (2), (3) have a rational solution which is perhaps not immediately obvious.

Our result is the following. Consider the quadratic program of finding a nonnegative m-vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ and a number μ such that

(2)* μ is a minimum

subject to

(3)* $$\mu = \left(\sum \lambda_i c_i\right)^2 + \sum \lambda_i\left(\rho_i^2 - c_i^2\right)$$

and

$$\sum \lambda_i = 1 \qquad \left(\sum \text{ is short for } \sum_{i=1}^{m}\right) .$$

We assert that (2), (3) and $(2)^*$, $(3)^*$ are dual problems. More precisely:

<u>Theorem</u>:

Let $(\bar{\lambda},\bar{\mu})$ solve $(2)^*$, $(3)^*$. Then $(\bar{x},\bar{\mu})$ solves (2), (3) where $\bar{x} = \sum \bar{\lambda}_i c_i$.

∎ We first prove the usual inequality (so called "weak duality"). Let (x,z) be any solution of (3) and (λ,μ) any solution of $(3)^*$. Multiplying (3) by λ_i and summing gives

$$0 \geqq \sum \lambda_i (x - c_i)^2 - \sum \lambda_i \rho_i^2 + z = x^2 - 2\left(\sum \lambda_i c_i\right)x + \sum \lambda_i\left(c_i^2 - \rho_i^2\right) + z$$

$$= \left(x - \sum \lambda_i c_i\right)^2 - \left(\sum \lambda_i c_i\right)^2 - \sum \lambda_i\left(\rho_i^2 - c_i^2\right) + z$$

$$= \left(x - \sum \lambda_i c_i\right)^2 + z - \mu \geqq z - \mu \text{ , (from } (3)^*)$$

so

(4) $$z \leqq \mu \text{ .}$$

Now suppose $(\bar{\lambda},\bar{\mu})$ is a solution of $(2)^*$, $(3)^*$, and let $\bar{x} = \sum \lambda_i c_i$. The Kuhn-Tucker conditions for this problem assert the existence of a number u such that

(5)
$$u \leq 2\bar{x}c_i + \rho_i^2 - c_i^2 \quad \text{for all } i \text{ and}$$
$$u = 2\bar{x}c_i + \rho_i^2 - c_i^2 \quad \text{if } \bar{\lambda}_i > 0 \text{ .}$$

Multiplying (5) by $\bar{\lambda}_i$ and summing gives

$$u = 2\bar{x}^2 + \sum \lambda_i\left(\rho_i^2 - c_i^2\right),$$

or

(6) $$u - \bar{x}^2 = \bar{x}^2 + \sum \lambda_i\left(\rho_i^2 - c_i^2\right) = \bar{\mu} \quad \text{from } (3)^*.$$

Subtracting $\bar{x}^2$ from (5) gives

$$u - \bar{x}^2 \leq \rho_i^2 - (\bar{x} - c_i)^2$$

so from (6)

$$(\bar{x} - c_i)^2 - \rho_i^2 + \bar{\mu} \leq 0$$

so $(\bar{x},\bar{\mu})$ is a solution of (3) and from (4) it is optimal, completing the proof.■

Remarks:

1. The fact that problems (2), (3) and $(2)^*$, $(3)^*$ are nonlinear duals can be derived by appealing to general nonlinear duality theorems, e.g., [2] and making some further simplifications. For this special case, however, we prefer the self-contained proof which uses only the standard optimality criterion (Kuhn-Tucker conditions) for quadratic programs.

2. One might hope to generalize the result of this note by replacing the spherical inequalities of (1) by general positive definite inequalities so that the spheres of (1) become ellipsoids. The following interesting example due to G. Bergman shows that there is no hope of doing this. The following inequalities describe ellipses in the plane.

$$
\begin{aligned}
2x^2 + 3xy + 2y^2 - 4x - 4y + 2 &\leqq 0 \\
2x^2 + 2xy + y^2 - 2x &\leqq 0 \\
x^2 + xy + y^2 - y &\leqq 0 .
\end{aligned}
$$

These inequalities have the unique solution $(2^{2/3} - 2^{1/3}, 2 - 2^{2/3})$.

3. The above example involves three inequalities. It seems possible that for the case of only two positive definite inequalities rational solutions will always exist. The answer is not known at this time.

BIBLIOGRAPHY

[1] Elzinga, D. J. and D. W. Hearn, "The Minimum Covering Sphere Problem," Management Science, Vol. 19, No. 1, pp. 96-140, (September 1972).

[2] Wolfe, P., "A Duality Theorem for Nonlinear Programming," Quarterly of Applied Mathematics, Vol. 19, pp. 239-244, (1961).